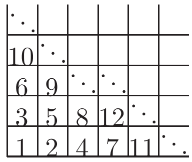
1-st Baltic Way

Riga, Latvia – November 24, 1990

1. Numbers $1, 2, \ldots, n$ are written around a circle in some order. What is the smallest possible sum of the absolute differences of adjacent numbers?

2. The squares of a squared paper are enumerated as shown on the picture. Devise a polynomial $p(m, n)$ in two variables such that for any $m, n \in \mathbb{N}$ the number written in the square with coordinates (m, n) is equal to $p(m, n)$.



3. Given $a_0 > 0$ and $c > 0$, the sequence (a_n) is defined by

$$a_{n+1} = \frac{a_n + c}{1 - ca_n} \quad \text{for } n = 0, 1, \ldots$$

Is it possible that $a_0, a_1, \ldots, a_{1989}$ are all positive but a_{1990} is negative?

4. Prove that, for any real numbers $a_1, a_2, \ldots, a_n$,

$$\sum_{i,j=1}^{n} \frac{a_i a_j}{i + j - 1} \geq 0.$$

5. Let $*$ be an operation, assigning a real number $a * b$ to each pair of real numbers (a, b). Find an equation which is true (for all possible values of variables) provided the operation $*$ is commutative or associative and which can be false otherwise.

6. Let $ABCD$ be a quadrilateral with $AD = BC$ and $\angle DAB + \angle ABC = 120^\circ$. An equilateral triangle DPC is erected in the exterior of the quadrilateral. Prove that the triangle APB is also equilateral.

7. The midpoint of each side of a convex pentagon is connected by a segment with the centroid of the triangle formed by the remaining three vertices of the pentagon. Prove that these five segments have a common point.

8. It is known that for any point P on the circumcircle of a triangle ABC, the orthogonal projections of P onto AB, BC, CA lie on a line, called a *Simson line* of P. Show that the Simson lines of two diametrically opposite points P_1 and P_2 are perpendicular.

9. Two congruent triangles are inscribed in an ellipse. Are they necessarily symmetric with respect to an axis or the center of the ellipse?

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10. A segment AB is marked on a line t. The segment is moved on the plane so that it remains parallel to t and that the traces of points A and B do not intersect. The segment finally returns onto t. How far can point A now be from its initial position?

11. Prove that the modulus of an integer root of a polynomial with integer coefficients cannot exceed the maximum of the moduli of the coefficients.

12. Let m and n be positive integers. Show that $25m + 3n$ is divisible by 83 if and only if so is $3m + 7n$.

13. Show that the equation $x^2 - 7y^2 = 1$ has infinitely many solutions in natural numbers.

14. Do there exist 1990 pairwise coprime positive integers such that all sums of two or more of these numbers are composite numbers?

15. Prove that none of the numbers $2^{2^n} + 1$, $n = 0, 1, 2, \dots$ is a perfect cube.

16. A closed polygonal line is drawn on a unit squared paper so that its vertices lie at lattice points and its sides have odd lengths. Prove that its number of sides is divisible by 4.

17. There are two piles with 72 and 30 candies. Two students alternate taking candies from one of the piles. Each time the number of candies taken from a pile must be a multiple of the number of candies in the other pile. Which student can always assure taking the last candy from one of the piles?

18. Numbers $1, 2, \dots, 101$ are written in the cells of a 101×101 square board so that each number is repeated 101 times. Prove that there exists either a column or a row containing at least 11 different numbers.

19. What is the largest possible number of subsets of the set $\{1, 2, \dots, 2n+1\}$ such that the intersection of any two subsets consists of one or several consecutive integers?

20. A creative task: propose an original competition problem together with its solution.