Mathematical Competition Baltic Way 2004

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Problems with Solutions

1 (EST) Given a sequence $a_1, a_2, a_3, \ldots$ of non-negative real numbers satisfying the conditions

1. $a_n + a_{2n} \geq 3n$,
2. $a_{n+1} + n \leq 2\sqrt{a_n \cdot (n+1)}$

for all indices $n = 1, 2, \ldots$.

(a) Prove that the inequality $a_n \geq n$ holds for every $n \in \mathbb{N}$.

(b) Give an example of such a sequence.

Solution:

(a) Note that the inequality

$$\frac{a_{n+1} + n}{2} \geq \sqrt{a_{n+1} \cdot n}$$

holds, which together with the second condition of the problem gives

$$\sqrt{a_{n+1} \cdot n} \leq \sqrt{a_n \cdot (n+1)}.$$

This inequality simplifies to

$$\frac{a_{n+1}}{a_n} \leq \frac{n+1}{n}.$$

Now, using the last inequality for the index n replaced by $n, n+1, \ldots, 2n-1$ and multiplying the results, we obtain

$$\frac{a_{2n}}{a_n} \leq \frac{2n}{n} = 2$$

or $2a_n \geq a_{2n}$. Taking into account the first condition of the problem, we have

$$3a_n = a_n + 2a_n \geq a_n + a_{2n} \geq 3n,$$

which implies $a_n \geq n$.

(b) The sequence $a_n = n + 1$ satisfies all the conditions of the problem.

2 (LAT) Let $P(x)$ be a polynomial with non-negative coefficients. Prove that if $P(\frac{1}{x})P(x) \geq 1$ for $x = 1$ then the same inequality holds for each positive x.

Solution: For $x > 0$ we have $P(x) > 0$. From the given condition we have $(P(1))^2 \geq 1$. Further, let's denote $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, then

$$P(x)P\left(\frac{1}{x}\right) = (a_0x^n + a_1x^{n-1} + \cdots + a_n)\left(a_0\left(\frac{1}{x}\right)^n + a_1\left(\frac{1}{x}\right)^{n-1} + \cdots + a_n\right) \geq$$

$$\text{(using the Couchy-Schwarz-Bunyakowski inequality)}$$

$$\geq \left(\sqrt{\frac{a_0}{x^n}}\sqrt{a_0x^n} + \sqrt{\frac{a_1}{x^{n-1}}}\sqrt{a_1x^{n-1}} + \cdots + \sqrt{a_n}\sqrt{a_n}\right)^2 \geq$$

$$\geq (a_0 + a_1 + \cdots + a_n)^2 = (P(1))^2 \geq 1.$$

3 (NOR) Let p, q, r be positive real numbers and $n \in \mathbb{N}$. Show that if $pqr = 1$, then

$$\frac{1}{p^n + q^n + 1} + \frac{1}{q^n + r^n + 1} + \frac{1}{r^n + p^n + 1} \leq 1.$$

Solution: The key idea is to deal with the case $n = 3$. Put $a = p^{n/3}$, $b = q^{n/3}$, and $c = r^{n/3}$, therefore $abc = (pqr)^{n/3} = 1$ and

$$\frac{1}{p^n + q^n + 1} + \frac{1}{p^n + r^n + 1} + \frac{1}{r^n + p^n + 1} =$$

$$= \frac{1}{a^3 + b^3 + 1} + \frac{1}{b^3 + c^3 + 1} + \frac{1}{c^3 + a^3 + 1}.$$

Now

$$\frac{1}{a^3 + b^3 + 1} = \frac{1}{(a + b)(a^2 - ab + b^2) + 1} =$$

$$= \frac{1}{(a + b)((a - b)^2 + ab) + 1} \leq \frac{1}{(a + b)ab + 1}.$$

Since $ab = c^{-1}$,

$$\frac{1}{a^3 + b^3 + 1} \leq \frac{1}{(a + b)ab + 1} = \frac{c}{a + b + c}.$$

Similarly we obtain

$$\frac{1}{b^3 + c^3 + 1} \leq \frac{a}{a + b + c} \quad \text{and} \quad \frac{1}{c^3 + a^3 + 1} \leq \frac{b}{a + b + c}.$$

Hence

$$\frac{1}{a^3 + b^3 + 1} + \frac{1}{b^3 + c^3 + 1} + \frac{1}{c^3 + a^3 + 1} \leq$$

$$\leq \frac{c}{a + b + c} + \frac{a}{a + b + c} + \frac{b}{a + b + c} = 1$$

And we are at home.

4 (LAT) Let $x_1, x_2, \ldots, x_n$ be real numbers with arithmetic mean X. Prove that there is a positive integer K such that the arithmetic mean of each of the lists $\{x_1, x_2, \ldots, x_K\}$, $\{x_2, x_3, \ldots, x_K\}$, $\{x_3, \ldots, x_K\}$, ..., $\{x_{K-1}, x_K\}$, $\{x_K\}$ is not greater than X.

Solution: Consider n bottles, each of volume 1 liter. Suppose they are filled with spirits of concentrations $x_1, x_2, \ldots, x_n$ (the measurement with spirits can be correspondingly chosen). The arithmetic mean of any set of $x_i - s$ is a concentration of corresponding mixture.

$$\underset{B_1}{\lfloor x_1 \rfloor} \quad \underset{B_2}{\lfloor x_2 \rfloor} \quad \underset{B_3}{\lfloor x_3 \rfloor} \ldots \underset{B_n}{\lfloor x_n \rfloor} .$$

Suppose the contrary to what must be proved. It means: for each bottle B_K such a bottle B_l, $l \leq K$, can be found that the concentration of a mixture of bottles $B_l, B_{l+1}, \ldots, B_K$ exceeds X.

Find such a B_{l_1} for B_n; B_{l_2} for B_{l_1-1}; ...; $B_{l_m} \equiv B_1$ for B_{l_m-1}. Consider the "segments"

$$\left[B_1 \ldots B_{l_{m-1}-1}\right]\left[B_{l_{m-1}} \ldots B_{l_{m-2}-1}\right] \ldots \left[B_{l_1} \ldots B_n\right].$$

In each of the segments the concentration of a mixture is greater than X; therefore the concentration of a mixture from all bottles is greater than X; a contradiction.

5 (DEN) Determine the range of the following function defined for integer k,

$$f(k) = (k)_3 + (2k)_5 + (3k)_7 - 6k ,$$

where $(k)_{2n+1}$ denotes the multiple of $2n+1$ closest to k.

Solution: For odd n we have

$$(k)_n = k + \frac{n-1}{2} - \left(k + \frac{n-1}{2}\right) \bmod n ,$$

where $m \bmod n$ denotes the principal remainder. Hence we get

$$f(k) = 6 - (k+1) \bmod 3 - (2k+2) \bmod 5 - (3k+3) \bmod 7 ,$$

The condition that the principal remainders take the values a, b and c, respectively, may be written

$$k+1 \equiv a \bmod 3 , \quad 2k+2 \equiv b \bmod 5 , \quad 3k+3 \equiv c \bmod 7 ,$$

or

$$k \equiv a-1 \bmod 3 , \quad k \equiv -2b-1 \bmod 5 , \quad k \equiv -2c-1 \bmod 7 .$$

By the Chinese Remainder Theorem, these congruences have a solution for any set of a, b and c. Hence $f(k)$ takes all the integer values between $6-2-4-6 = -6$ and $6-0-0-0 = 6$.

6 (SWE) A positive integer is written on each of the six faces of a cube. For each vertex of the cube we compute the product of the numbers on the three adjacent faces. The sum of these products is 1001. What is the sum of the six numbers on the faces?

Solution: Let the numbers on the faces be $a_1, a_2, b_1, b_2, c_1, c_2$, placed in such a way that a_1 and a_2 are on opposite faces etc. Then the sum of the eight products is equal to $(a_1 + a_2)(b_1 + b_2)(c_1 + c_2) = 1001 = 7 \cdot 11 \cdot 13$. Hence the sum of the numbers on the faces is $a_1 + a_2 + b_1 + b_2 + c_1 + c_2 = 7 + 11 + 13 = 31$.

7 (EST) Find all sets X consisting of at least two positive integers such that for every pair $m, n \in X$, where $n > m$, there exists $k \in X$ such that $n = mk^2$.

Solution: Answer: All the sets $\{m, m^3\}$, where $m > 1$.

Let X be a set satisfying the condition of the problem and let m and n, where $n > m$, be the two smallest elements in the set X. There has to exist a $k \in X$ so that $n = mk^2$, but as $m \leq k \leq n$, either $k = n$ or $k = m$. The first case gives $m = n = 1$, a contradiction; the second case implies $n = m^3$ with $m > 1$.

Suppose there exists the third smallest element $q \in X$. Then there also exists $k_0 \in X$, such that $q = mk_0^2$. We have $q > k_0 \geq m$, but $k_0 = m$ would imply $q = n$, thus $k_0 = n = m^3$ and $q = m^7$. Now for n and q there has to exist $k_1 \in X$ such that $q = nk_1^2$, which gives $k_1 = m^2$. Since $m^2 \notin X$, we have a contradiction.

Thus we see that the only elements that the set X can contain, are m and m^3 for some $m > 1$. One can easily verify that all such sets $X = \{m, m^3\}$ satisfy the conditions of the problem.

8 (NOR) Let $f(x)$ be a non-constant polynomial with integer coefficients. Prove that there is an integer n such that $f(n)$ has at least 2004 distinct prime factors.

Solution: Suppose the contrary. Choose an integer n_0 so that $f(n_0)$ has the highest number of prime factors. By translating the polynomial we may assume

$n_0 = 0$. Setting $k = f(0)$, we have $f(wk^2) \equiv k \bmod k^2$, or $f(wk^2) = ak^2 + k = (ak+1)k$. Since $gcd(ak+1, k) = 1$ and k alone achieves the highest number of prime factors of f, we must have $ak + 1 = \pm 1$. This cannot happen for every w since f is non-constant, so we have a contradiction.

Alternative solution: Let $S = \{p | p \text{ is prime and divides } f(n) \text{ for some } n \in \mathbb{Z}\}$. If $|S| \geq 2004$, choose primes $p_1, p_2, ..., p_{2004}$ and integers $a_1, a_2, ..., a_{2004}$ so that $f(a_i) \equiv 0 \bmod p_i$. By the Chinese Remainder Theorem, there is an x so that $x \equiv a_i \bmod p_i$ for all i. This x satisfies $p_1, p_2, ..., p_{2004} | f(x)$. Otherwise, if $|S| < 2004$, let $A = \{m | (p | m \Rightarrow p \in S)\} \cup \{0\}$. $m \in A$ can be written $m = \pm \prod_{i=1}^{r} p_i^{e_i}$ with $r = |S| < 2004$, $0 \leq e_i \leq \log_{p_i} m \leq \log_2 m \Rightarrow |A \cap [-N, N]| \leq 2(\log N + 1)^{2003} + 1$. Let d be the degree of f, then there is an $c > 0$ so that $|f(Z) \cap [-N, N]| \geq c\sqrt[d]{N}$, which grows asymptotically faster than $2(\log N + 1)^{2003} + 1$ and thus contradicts the assumption that $|S| < 2004$.

9 (EST) A set S of $n-1$ natural numbers is given ($n \geq 3$). There exist at least two elements in this set whose difference is not divisible by n. Prove that it is possible to choose a non-empty subset of S so that the sum of its elements is divisible by n.

Solution: Suppose to the contrary that there exists a set $X = \{a_1, a_2, \ldots, a_{n-1}\}$ violating the statement of the problem, and let $a_{n-2} \not\equiv a_{n-1} \bmod n$. Denote $S_i = a_1 + a_2 + \ldots + a_i$, $i = 1, \ldots, n-1$. The conditions of the problem imply that all the numbers S_i must give different remainders when divided by n. Indeed, if for some $j < k$ we had $S_j \equiv S_k \bmod n$, then $a_{j+1} + a_{j+2} + \ldots + a_k \equiv S_k - S_j \equiv 0 \bmod n$. Consider now the sum $S' = S_{n-3} + a_{n-1}$. We see that S' can not be congruent to any of the sums S_i (for $i \neq n-2$ the above argument works and for $i = n-2$ we use the assumption $a_{n-2} \not\equiv a_{n-1} \bmod n$). Thus we have n sums that give pairwise different remainders when divided by n, consequently one of them has to give the remainder 0, a contradiction.

10 (LAT) Is there an infinite sequence of prime numbers $p_1, p_2, \ldots, p_n, p_{n+1}, \ldots$ such that $|p_{n+1} - 2p_n| = 1$ for each $n \in \mathbb{N}$?

Solution: Answer. No, there is no such sequence.

Suppose the contrary. Clearly $p_3 > 3$. We have two possibilities:

a) $p_3 \equiv 1 (\bmod 3)$. Then obligatory $p_4 = 2p_3 - 1$ (otherwise $p_4 \equiv 0 (\bmod 3)$, so $p_4 \equiv 1 (\bmod 3)$. Analogously $p_5 = 2p_4 - 1$, $p_6 = 2p_5 - 1$ etc. By an easy induction we have

$$p_{n+1} - 1 = 2^{n-2}(p_3 - 1), \qquad n = 3; 4; 5; \ldots.$$

If we set $n = p_3 + 1$ we have $p_{p_3+2} - 1 = 2^{p_3 - 1}(p_3 - 1)$, from which

$$p_{p_3+2} \underset{\bmod p_3}{\equiv} 1 + 1 \cdot (p_3 - 1) = p_3 \underset{\bmod p_3}{\equiv} 0$$

– a contradiction.

b) $p_3 \equiv 2 (\bmod 3)$ is treated analogously.

11 (BLR) Given a table $m \times n$, in each cell of which a number $+1$ or -1 is written. It is known that initially exactly one -1 is in the table, all the other numbers being $+1$. During a move, it is allowed to chose any cell containing -1, replace this -1 by 0, and simultaneously multiply all the numbers in the neighboring cells by -1 (we say that two cells are neighboring if they have a common side). Find all (m, n) for which using such moves one can obtain the table containing zeros only, regardless of the cell in which the initial -1 stands.

Solution: Answer: those (m, n) for which at least one of m, n is odd.

Let us erase a unit segment which is common side of some two cells any time when in both the cells two zeros appears. If the final table consists of zeros only, all the unit segments (except those which belong to the border of the table) are erased. We must erase, in common,

$$m(n-1) + n(m-1) = 2mn - m - n$$

such unit segments.

On the other hand, in order to obtain 0 in a cell with initial $+1$ one must first obtain -1 in this cell, that is the sign of the number in this cell must be changed, in common, an odd number of times (namely, 1 or 3). Hence, any cell with -1 (except the initial one) has an odd number of neighboring zeros. So, any time we replace -1 by 0 we erase an odd number of unit segments. That is, the total number of unit segments is congruent modulo 2 to the initial number of $+1$'s in the table. Therefore

$$2mn - m - n \equiv mn - 1 \pmod 2 \quad \Longrightarrow \quad (m-1)(n-1) \equiv 0 \pmod 2,$$

so, at least one of m, n is odd.

It remains to show that if, for example, n is odd, we can obtain a zero table.

First, if -1 is in the i-th row, we can obtain all zeros in this i-th row (see, for example, Fig.1, for $n = 7$, where initially -1 stands in the shaded cell. Numbers 1,2,... written in the row show the sequence of the moves; that is,in the k-th move we replace -1 by 0 in the cell where this k is written).

Now, having the row of -1's and performing the sequence of moves as shown in the Fig. 2, we obtain a table with zeros in this row and -1's in the neighboring row (here we first replace -1's by 0's in the odd positions of the row, then replace the remaining -1's by 0's in the even positions). Continuing in this way we finally obtain a zero table.

3	2	1	4	5	6	7

$\longrightarrow$

−1	−1	−1	−1	−1	−1	−1
0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	−1

Fig. 1

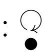
−1	−1	−1	−1	−1	−1	−1
0	0	0	0	0	0	0
1	3	2	5	4	7	6

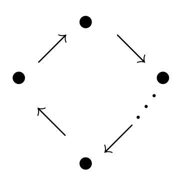
$\longrightarrow$

−1	−1	−1	−1	−1	−1	−1
0	0	0	0	0	0	0
0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	−1

Fig. 2

12 (LAT) There are $2n$ different numbers in a row. By one move we can interchange any two numbers or interchange any 3 numbers cyclically (choose a, b, c and place a instead of b, b instead of c, and c instead of a). What is the minimal number of moves that is always sufficient to arrange the numbers in increasing order?

Solution: If number y occupies the place where x should be at the end, we draw an arrow $x \to y$. Clearly at the beginning all numbers are arranged in several cycles:  (a loop),  (binary cycle),



(at least 3 numbers; "long"cycle). Our aim is to obtain $2n$ loops.

Clearly each binary cycle can be rearranged into 2 loops by one move. If there is a long cycle with a fragment $\dots \underset{a}{\bullet} \longrightarrow \underset{b}{\bullet} \longrightarrow \underset{c}{\bullet} \dots$, interchange a, b, c cyclically; at least 2 loops  and  appear. Go on if necessary. By each move the number of loops increase by 2, so at most n moves are needed.

Comment. The problem can be made considerably more difficult by asking for minimal number of moves that is always enough. By examining all possible cases how the interchanged numbers can be distributed among disjoint cycles we easily see that the number of disjoint cycles increases by at most 2 per one move. So, if there is one cycle at the beginning, we can not get $2n$ loops in less than n moves.

13 (FIN) The 25 member states of the European Union set up a committee with the following rules: 1) the committee should meet daily; 2) at each meeting, at least one member state should be represented; 3) at any two different meetings, a different set of member states should be represented; and 4) at the n^{th} meeting, for every $k < n$, the set of states represented should include at least one state that was represented at the k^{th} meeting. For how many days can the committee have its meetings?

Solution: If one member is always represented, rules 2 and 4 will be fulfilled. There are 2^{24} different subsets of the remaining 24 members. So there can be at least 2^{24} meetings. Rule 3 forbids complimentary sets in two different meetings. So the maximal number of meetings cannot exceed $\frac{1}{2} \cdot 2^{25}$. So the maximal number of meetings for the commission is exactly $2^{24} = 16777216$.

14 (SPB) We say that a pile is a set of four or more nuts. Two persons play the following game. They start with one pile of $n \geq 4$ nuts. During a move a player takes one of the piles that they have and split it into two nonempty sets (these sets are not necessarily piles, they can contain arbitrary number of nuts). If the player cannot move, he loses. For which values of n does the first player have a winning strategy?

Solution: Answer: First player wins.

More over, if the pile in the beginning of the game contains $4k$, $4k+1$, $4k+2$ nuts, then the first player wins, and if the pile contains $4k+3$ ($k \geq 1$) nuts, then the second player wins.

The set of 1, 2 or 3 we will call a degenerate pile.

We will prove the answer by induction on k together with the *useful fact* that in the position consisting of two piles of $4t+1$ and $4s+1$ nuts (where $s+t \leq k$; $s, t \geq 1$) the second player wins. We omit here the base of induction.

So let us prove the step of induction. Assume that we know the answer for piles with at most $4k-1$ nuts and the useful fact for $s+t \leq k$ nuts, and prove the answer for piles with $4k$, $4k+1$, $4k+2$, $4k+3$ nuts and the useful fact for $s+t \leq k+1$ nuts.

1. If the pile contains $4k$, $4k+1$ or $4k+2$ nuts, then the first player wins: he "cuts off" 1, 2 or 3 nuts, obtain a pile of $4k-1$ nuts and therefore wins by induction hypothesis.

2. Let the pile contain $4k+3$ nuts. The first player can split it in two different ways: $(4\ell+1; 4m+2)$ or $(4\ell; 4m+3)$. In the first case the second player wins directly if one of the piles is degenerate (i.e. contains at most 3 nuts) due to the previous paragraph. Otherwise, he removes one nut from the second pile and wins due to the useful fact. In the second case he removes one nut from the first pile and wins because in the position $(4\ell-1, 4m+3)$ he can use in both piles the winning strategy of the second player. (The case of degenerate piles here is trivial.)

3. Now prove that in the position $(4t+1; 4s+1)$ where $s+t = k+1$, the second player wins. Due to the symmetry we can think that the first player splits the second pile. As in the previous paragraph we have two possible moves: $(4t+1; 4u; 4v+1)$ and $(4t+1; 4u+2; 4v+3)$.

3a) Consider a case $(4t+1; 4u; 4v+1)$. If $u = 1$, the second player moves $4u = 4 \to (2; 2)$ and wins by the useful fact because actually he has obtained a position $(4t+1; 4v+1)$.

Otherwise he moves $4u \to (1; 4u-1)$ and wins because he has winning strategies in both positions $(4t+1; 4v+1)$ and $4u-1$.

3b) Consider a case $(4t+1; 4u+2; 4v+3)$. If $u = 0$, the second player moves $4v + 3 \to (2; 4v+1)$ and wins by the useful fact because actually he has obtained a position $(4t+1; 4v+1)$. Otherwise he moves $4u+2 \to (1; 4u+1)$ and wins because he has winning strategies in both positions $(4t+1; 4u+1)$ and $4v+3$.

15 (SWE) A circle is divided into 13 segments, numbered consecutively from 1 to 13. Five fleas called A, B, C, D and E are sitting in the segments 1, 2, 3, 4 and 5. A flea is allowed to jump to an empty segment five positions away in either direction around the circle. Only one flea jumps at the same time, and two fleas cannot be in the same segment. After some jumps, the fleas are back in the segments 1,2,3,4,5, but possibly in some other order than they started. Which orders are possible?

Solution: Write the numbers from 1 to 13 in the order **1**, 6, 11, **3**, 8, 13, **5**, 10, **2**, 7, 12, **4**, 9. Then each time a flea jumps it moves between two adjacent numbers or between the first and the last in this row. Since a flea can never move past another flea, the possible permutations are

1	3	5	2	4
A	C	E	B	D
D	A	C	E	B
B	D	A	C	E
E	B	D	A	C
C	E	B	D	A

or equivalently

1	2	3	4	5
A	B	C	D	E
D	E	A	B	C
B	C	D	E	A
E	A	B	C	D
C	D	E	A	B

16 (DEN) Through a point P exterior to a given circle pass a secant and a tangent to the circle. The secant intersects the circle at A and B, and the tangent touches the circle at C on the same side of the diameter through P as A and B. The projection of C on the diameter is Q. Prove that QC bisects $\angle AQB$.

Solution: Denoting the center of the circle by O, we have $OQ \times OP = OA^2 = OB^2$. Hence $\triangle OAQ \sim \triangle OPA$ and $\triangle OBQ \sim \triangle OPB$. Since $\triangle AOB$ is isosceles, we have $\angle OAP + \angle OBP = 180°$, and therefore

$$\begin{aligned} \angle AQP + \angle BQP &= \angle AOP + \angle OAQ + \angle BOP + \angle OBQ \\ &= \angle AOP + \angle OPA + \angle BOP + \angle OPB \\ &= 180° - \angle OAP + 180° - \angle OBP = 180°\,. \end{aligned}$$

Thus QC, being perpendicular to QP, bisects $\angle AQB$.

17 (SWE) Consider a rectangle with side lengths 3 and 4, and pick an arbitrary inner point on each side. Let x, y, z and u denote the side lengths of the quadrilateral spanned by these points. Prove that $25 \le x^2 + y^2 + z^2 + u^2 \le 50$.

Solution: Let a, b, c, and d be the distances of the chosen points from the midpoints of the sides of the rectangle (with a and c on the sides of length 3). Then

$$\begin{aligned} x^2+y^2+z^2+u^2 &= \left(\frac{3}{2}+a\right)^2 + \left(\frac{3}{2}-a\right)^2 + \left(\frac{3}{2}+c\right)^2 + \left(\frac{3}{2}-c\right)^2 + \\ &+ (2+b)^2 + (2-b)^2 + (2+d)^2 + (2-d)^2 = \\ &= 4\cdot\left(\frac{3}{2}\right)^2 + 4\cdot 2^2 + 2(a^2+b^2+c^2+d^2) = \\ &= 25 + 2(a^2+b^2+c^2+d^2). \end{aligned}$$

Since $0 \le a^2 \le (3/2)^2$, $0 \le c^2 \le (3/2)^2$, $0 \le b^2 \le 2^2$ and $0 \le d^2 \le 2^2$ the desired inequalities follow.

18 (LAT) A ray emanating from the vertex A of the triangle ABC intersects the side BC at X and the circumcircle of ABC at Y. Prove that $\frac{1}{AX} + \frac{1}{XY} \ge \frac{4}{BC}$.

Solution: From the geometric mean – harmonic mean inequality we have

$$\frac{1}{AX} + \frac{1}{XY} \ge \frac{2}{\sqrt{AX \cdot XY}}. \tag{1}$$

As BC and AY are chords intersecting at X we have $AX \cdot XY = BX \cdot XC$. Therefore (1) transforms into

$$\frac{1}{AX} + \frac{1}{XY} \ge \frac{2}{\sqrt{BX \cdot XC}}. \tag{2}$$

After all, $\sqrt{BX \cdot XC} \le \frac{BX+XC}{2} = \frac{BC}{2}$. So from (2) we obtain what is needed.

19 (SPB) D is the midpoint of the side BC of the given triangle ABC. M is a point on the side BC such that $\angle BAM = \angle DAC$. L is the second intersection point of the circumcircle of the triangle CAM with the side AB. K is the second intersection point of the circumcircle of the triangle BAM with the side AC. Prove that $KL \parallel BC$.

Solution: It is sufficient to prove that $CK : LB = AC : AB$.

The triangles ABC and MKC are similar because they have common angle C and $\angle CMK = 180^\circ - \angle BMK = \angle KAB$ (the latter equality is due to the observation that $\angle BMK$ and $\angle KAB$ are the opposite angles in the inscribed quadrilateral $AKMB$).

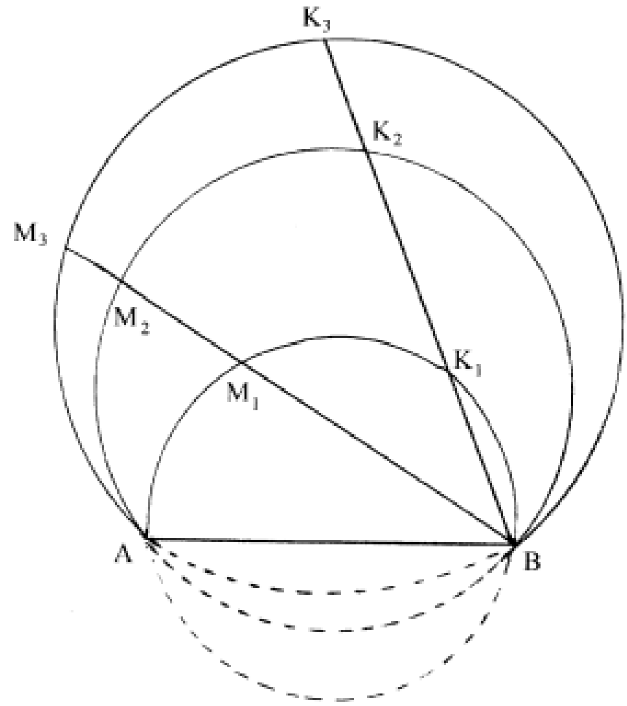
By the analogous reasons the triangles ABC and MBL are similar. Therefore the triangles MKC and MBL are also similar and we have

$$\frac{CK}{LB} = \frac{KM}{BM} \underset{(1)}{=} \frac{\frac{AM \sin KAM}{\sin AKM}}{\frac{AM \sin MAB}{\sin MBA}} \underset{(2)}{=} \frac{\sin KAM}{\sin MAB} \underset{(3)}{=} \frac{\sin DAB}{\sin DAC} \underset{(4)}{=} \frac{\frac{BD \sin BDA}{AB}}{\frac{CD \sin CDA}{AC}} = \frac{AC}{AB}.$$

(1) is due to sinus theorem for triangles AKM and ABM; (2) is due to the equality $\angle AKM = 180^\circ - \angle MBA$ in the inscribed quadrilateral $AKMB$; (3) is due to the definition of the point M; (4) is due to sinus theorem for triangles ACD and ABD;

20 (LAT) Three circular arcs w_1, w_2, w_3 with common endpoints A and B are on the same side of the line AB; w_2 lies between w_1 and w_3. Two rays emanating from B intersect these arcs at M_1, M_2, M_3 and K_1, K_2, K_3, respectively. Prove that $\frac{M_1M_2}{M_2M_3} = \frac{K_1K_2}{K_2K_3}$.

Solution:



From inscribed angles we have $\angle AK_1B = \angle AM_1B$, $\angle AK_2B = \angle AM_2B$. From this it follows that $\triangle AK_1K_2 \sim \triangle AM_1M_2$, so

$$\frac{K_1K_2}{M_1M_2} = \frac{AK_2}{AM_2}\,. \tag{1}$$

Similarly $\triangle AK_2K_3 \sim \triangle AM_2M_3$, so

$$\frac{K_2K_3}{M_2M_3} = \frac{AK_2}{AM_2}\,. \tag{2}$$

From (1) and (2) we get $\frac{K_1K_2}{M_1M_2} = \frac{K_2K_3}{M_2M_3}$, from which desired property follows.

Comment. A Thaler theorem for circles, isn't it?