41st International Mathematical Olympiad

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Problems

Problem 1

$A\,B$ is tangent to the circles $C\,A\,M\,N$ and $N\,M\,B\,D$. M lies between C and D on the line $C\,D$, and $C\,D$ is parallel to $A\,B$. The chords $N\,A$ and $C\,M$ meet at P; the chords $N\,B$ and $M\,D$ meet at Q. The rays $C\,A$ and $D\,B$ meet at E. Prove that $P\,E = Q\,E$

Problem 2

A, B, C are positive reals with product 1. Prove that $(A - 1 + 1/B)(B - 1 + 1/C)(C - 1 + 1/A) \leq 1$.

Problem 3

k is a positive real. N is an integer greater than 1. N points are placed on a line, not all coincident. A *move* is carried out as follows. Pick any two points A and B which are not coincident. Suppose that A lies to the right of B. Replace B by another point B' to the right of A such that $A\,B' = k\,B\,A$. For what values of k can we move the points arbitrarily far to the right by repeated moves?

Problem 4

100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?

Problem 5

Can we find N divisible by just 2000 different primes, so that N divides $2^N + 1$? [N may be divisible by a prime power.]

Problem 6

$A_1\,A_2\,A_3$ is an acute-angled triangle. The foot of the altitude from A_i is K_i and the incircle touches the side opposite A_i at L_i. The line $K_1\,K_2$ is reflected in the line $L_1\,L_2$. Similarly, the line $K_2\,K_3$ is reflected in $L_2\,L_3$ and $K_3\,K_1$ is reflected in $L_3\,L_1$. Show that the three new lines form a triangle with vertices on the incircle.

Problems with Solutions

Problem 1

$A B$ is tangent to the circles $C A M N$ and $N M B D$. M lies between C and D on the line $C D$, and $C D$ is parallel to $A B$. The chords $N A$ and $C M$ meet at P; the chords $N B$ and $M D$ meet at Q. The rays $C A$ and $D B$ meet at E. Prove that $P E = Q E$

Solution

Angle $E B A$ = angle $B D M$ (because `C D` is parallel to `A B`) = angle $A B M$ (because $A B$ is tangent at B). So $A B$ bisects $E B M$. Similarly, $B A$ bisects angle $E A M$. Hence E is the reflection of M in $A B$. So $E M$ is perpendicular to $A B$ and hence to $C D$. So it suffices to show that $M P = M Q$.

Let the ray $N M$ meet $A B$ at X. $X A$ is a tangent so $X A^2 = X M . X N$. Similarly, $X B$ is a tangent, so $X B 2 = X M . X N$. Hence $X A = X B$. But $A B$ and $P Q$ are parallel, so $M P = M Q$.

Problem 2

A, B, C are positive reals with product 1. Prove that $(A - 1 + 1/B)(B - 1 + 1/C)(C - 1 + 1/A) \leq 1$.

Solution

An elegant solution due to Robin Chapman is as follows:

$(B - 1 + 1/C) = B(1 - 1/B + 1/(B C)) = B(1 + A - 1/B)$. Hence,
$(A - 1 + 1/B)(B - 1 + 1/C) = B(A 2 - (1 - 1/B) 2) \leq B\ A 2$. So the square of the product of all three $\leq B\ A 2\ C\ B 2\ A\ C 2 = 1$.

Actually, that is not quite true. The last sentence would not follow if we had some negative left hand sides, because then we could not multiply the inequalities. But it is easy to deal separately with the case where $(A - 1 + 1/B)$, $(B - 1 + 1/C)$, $(C - 1 + 1/A)$ are not all positive. If one of the three terms is negative, then the other two must be positive. For example, if $A - 1 + 1/B < 0$, then $A < 1$, so $C - 1 + 1/A > 0$, and $B > 1$, so $B - 1 + 1/C > 0$. But if one term is negative and two are positive, then their product is negative and hence less than 1.

Few people would manage this under exam conditions, but there are plenty of longer and easier to find solutions!

Problem 3

k is a positive real. N is an integer greater than 1. N points are placed on a line, not all coincident. A *move* is carried out as follows. Pick any two points A and B which are not coincident. Suppose that A lies to the right of B. Replace B by another point B' to the right of A such that $A B' = k B A$. For what values of k can we move the points arbitrarily far to the right by repeated moves?

Solution

Answer: $k \geq 1/(n - 1)$.

An elegant solution by Gerhard Woeginger is as follows:

Suppose $k < 1/(n - 1)$, so that $k_0 = 1/k - (n - 1) > 0$. Let X be the sum of the distances of the points from the rightmost point. If a move does not change the rightmost point, then it reduces X. If it moves the rightmost point a distance z to the right, then it reduces X by at least $z/k - (n - 1) z = k_0 z$. X cannot be reduced below nil. So the total distance moved by the rightmost point is at most X_0 / k_0, where X_0 is the initial value of X.

Conversely, suppose $k \geq 1/(n-1)$, so that $k_1 = (n-1) - 1/k \geq 0$. We always move the leftmost point. This has the effect of moving the rightmost point $z > 0$ and increasing X by $(n-1)z - z/k = k_1 z \geq 0$. So X is never decreased. But $z >= k X/(n-1) >= k X_0/(n-1) > 0$. So we can move the rightmost point arbitarily far to the right (and hence all the points, since another $n-1$ moves will move the other points to the right of the rightmost point).

Problem 4

100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?

Solution

Answer: 12. Place $1, 2, 3$ in different boxes (6 possibilities) and then place n in the same box as its residue mod 3. Or place 1 and 100 in different boxes and 2–99 in the third box (6 possibilities).

An *elegant* solution communicated (in outline) by both Mohd Suhaimi Ramly and Fokko J van de Bult is as follows:

Let H_n be the corresponding result that for cards numbered 1 to n the only solutions are by residue mod 3, or 1 and n in separate boxes and 2 to $n-1$ in the third box. It is easy to check that they *are* solutions. H_n is the assertion that there are no others. H_3 is obviously true (although the two cases coincide). We now use induction on n. So suppose that the result is true for n and consider the case $n+1$.

Suppose $n+1$ is alone in its box. If 1 is not also alone, then let N be the sum of the largest cards in each of the boxes not containing $n+1$. Since $n+2 \leq N \leq n+(n-1) = 2n-1$, we can achieve the same sum N as from a different pair of boxes as $(n+1)+(N-n-1)$. Contradiction. So 1 must be alone and we have one of the solutions envisaged in H_{n+1}.

If $n+1$ is not alone, then if we remove it, we must have a solution for n. But that solution cannot be the n, 1, 2 to $n-1$ solution. For we can easily check that none of the three boxes will then accomodate $n+1$. So it must be the mod 3 solution. We can easily check that in this case $n+1$ must go in the box with matching residue, which makes the $(n+1)$ solution the other solution envisaged by H_{n+1}. That completes the induction.

My much more plodding solution (which I was quite pleased with until I saw the more elegant solution above) follows. It took about half-an-hour and shows the kind of kludge one is likely to come up with under time pressure in an exam!

With a suitable labeling of the boxes as A, B, C, there are 4 cases to consider:

Case 1: A contains 1; B contains 2; C contains 3

Case 2: A contains 1, 2

Case 3: A contains 1, 3; B contains 2

Case 4: A contains 1; B contains 2, 3.

We show that Cases 1 and 4 each yield just one possible arrangement and Cases 2 and 3 none.

In Case 1, it is an easy induction that n must be placed in the same box as its residue (in other words numbers with residue 1 mod 3 go into A, numbers with residue 2 go into B, and numbers with residue 0 go into C). For $(n+1)+(n-2) = n+(n-1)$. Hence $n+1$ must go in the same box as $n-2$ (if they were in different boxes, then we would have two pairs from different pairs of boxes with the same sum). It is also clear that this is a possible arrangement. Given the sum of two numbers from different boxes, take its residue mod 3. A residue of 0 indicates that the third (unused) box was C, a residue of 1 indicates that the third box was A, and a residue of 2 indicates that the third box was B. Note that this unique arrangement gives 6 ways for the question, because there are 6 ways of arranging 1, 2 and 3 in the given boxes.

In Case 2, let n be the smallest number not in box A. Suppose it is in box B. Let m be the smallest number in the third box, C. $m-1$ cannot be in C. If it is in A, then $m+(n-1)=(m-1)+n$. Contradiction (m is in C, $n-1$ is in A, so that pair identifies B as the third box, but $m-1$ is in A and n is in B, identifying C). So $m-1$ must be in B. But $(m-1)+2=m+1$. Contradiction. So Case 2 is not possible.

In Case 3, let n be the smallest number in box C, so $n-1$ must be in A or B. If $n-1$ is in A, then $(n-1)+2=n+2$. Contradiction (a sum of numbers in A and B equals a sum from C and A). If $n-1$ is in B, then $(n-1)+3=n+2$. Contradiction (a sum from B and A equals a sum from C and B). So Case 3 is not possible.

In Case 4, let n be the smallest number in box C. $n-1$ cannot be in A, or $(n-1)+2=3+n$ (pair from A, B with same sum as pair from B, C), so $n-1$ must be in B. Now $n+1$ cannot be in A (or $(n+1)+2=3+n$), or in B or C (or $1+(n+1)=2+n$). So $n+1$ cannot exist and hence $n=100$. It is now an easy induction that all of $4, 5, \ldots 98$ must be in B. For given that m is in B, if $m+1$ were in A, we would have $100+m=99+(m+1)$. But this arrangement (1 in A, 2–99 in B, 100 in C) is certainly possible: sums 3–100 identify C as the third box, sum 101 identifies B as the third box, and sums 102–199 identify A as the third box. Finally, as in Case 1, this unique arrangement corresponds to 6 ways of arranging the cards in the given boxes.

Problem 5

Can we find N divisible by just 2000 different primes, so that N divides 2^N+1? [N may be divisible by a prime power.]

Solution

Answer: yes.

Note that for b odd we have $2^{ab}+1=(2^a+1)(2^{a(b-1)}-2^{a(b-2)}+\ldots+1)$, and so 2^a+1 is a factor of $2^{ab}+1$. It is sufficient therefore to find m such that (1) m has only a few distinct prime factors, (2) 2^m+1 has a large number of distinct prime factors, (3) m divides 2^m+1. For then we can take k, a product of enough distinct primes dividing 2^m+1 (but not m), so that km has exactly 2000 factors. Then km still divides 2^m+1 and hence $2^{km}+1$.

The simplest case is where m has only one distinct prime factor p, in other words it is a power of p. But if p is a prime, then p divides 2^p-2, so the only p for which p divides 2^p+1 is 3. So the questions are whether $a_h=2^m+1$ is (1) divisible by $m=3^h$ and (2) has a large number of distinct prime factors.

$a_{h+1}=a_h(2^{2m}-2^m+1)$, where $m=3^h$. But $2^m=(a_h-1)$, so $a_{h+1}=a_h(a_h^2-3a_h+3)$. Now $a_1=9$, so an easy induction shows that 3^{h+1} divides a_h, which answers (1) affirmatively. Also, since a_h is a factor of a_{h+1}, any prime dividing a_h also divides a_{h+1}. Put $a_h=3^{h+1}b_h$. Then $b_{h+1}=b_h(3^{2h+1}b_h^2-3^{h+2}b_h+1)$. Now $(3^{2h+1}b_h^2-3^{h+2}b_h+1)>1$, so it must have some prime factor $p>1$. But p be 3 or divide b_h (since $(3^{2h+1}b_h^2-3^{h+2}b_h+1)$ is a multiple of $3b_h$ plus 1), so b_{h+1} has at least one prime factor $p>3$ which does not divide b_h. So b_{h+1} has at least h distinct prime factors greater than 3, which answers (2) affirmatively. But that is all we need. We can take m in the first paragraph above to be 3^{2000}: (1) m has only one distinct prime factor, (2) $2^m+1=3^{2001}b_{2000}$ has at least 1999 distinct prime factors other than 3, (3) m divides 2^m+1. Take k to be a product of 1999 distinct prime factors dividing b_{2000}. Then $N=km$ is the required number with exactly 2000 distinct prime factors which divides 2^N+1.

Problem 6

$A_1A_2A_3$ is an acute-angled triangle. The foot of the altitude from A_i is K_i and the incircle touches the side opposite A_i at L_i. The line K_1K_2 is reflected in the line L_1L_2. Similarly, the line K_2K_3 is reflected in L_2L_3 and K_3K_1 is reflected in L_3L_1. Show that the three new lines form a triangle with vertices on the incircle.

Solution

Let O be the centre of the incircle. Let the line parallel to $A_1 A_2$ through L_2 meet the line $A_2 O$ at X. We will show that X is the reflection of K_2 in $L_2 L_3$. Let $A_1 A_3$ meet the line $A_2 O$ at B_2. Now $A_2 K_2$ is perpendicular to $K_2 B_2$ and $O L_2$ is perpendicular to $L_2 B_2$, so $A_2 K_2 B_2$ and $O L_2 B_2$ are similar. Hence $K_2 L_2 / L_2 B_2 = A_2 O / O B_2$. But $O A_3$ is the angle bisector in the triangle $A_2 A_3 B_2$, so $A_2 O / O B_2 = A_2 A_3 / B_2 A_3$.

Take B_2' on the line $A_2 O$ such that $L_2 B_2 = L_2 B_2'$ (B_2' is distinct from B_2 unless $L_2 B_2$ is perpendicular to the line). Then angle $L_2 B_2' X$ = angle $A_3 B_2 A_2$. Also, since $L_2 X$ is parallel to $A_2 A_1$, angle $L_2 X B_2'$ = angle $A_3 A_2 B_2$. So the triangles $L_2 X B_2'$ and $A_3 A_2 B_2$ are similar. Hence $A_2 A_3 / B_2 A_3 = X L_2 / B_2' L_2 = X L_2 / B_2 L_2$ (since $B_2' L_2 = B_2 L_2$).

Thus we have shown that $K_2 L_2 / L_2 B_2 = X L_2 / B_2 L_2$ and hence that $K_2 L_2 = X L_2$. $L_2 X$ is parallel to $A_2 A_1$ so angle $A_2 A_1 A_3$ = angle $A_1 L_2 X$ = angle $L_2 X K_2$ + angle $L_2 K_2 X$ = 2 angle $L_2 X K_2$ (isosceles). So angle $L_2 X K_2 = 1/2$ angle $A_2 A_1 A_3$ = angle $A_2 A_1 O$. $L_2 X$ and $A_2 A_1$ are parallel, so $K_2 X$ and $O A_1$ are parallel. But $O A_1$ is perpendicular to $L_2 L_3$, so $K_2 X$ is also perpendicular to $L_2 L_3$ and hence X is the reflection of K_2 in $L_2 L_3$.

Now the angle $K_3 K_2 A_1$ = angle $A_1 A_2 A_3$, because it is
$90 -$ angle $K_3 K_2 A_2 = 90 -$ angle $K_3 A_3 A_2$ ($A_2 A_3 K_2 K_3$ is cyclic with $A_2 A_3$ *a* diameter) = angle $A_1 A_2 A_3$.

So the reflection of $K_2 K_3$ in $L_2 L_3$ is a line through X making an angle $A_1 A_2 A_3$ with $L_2 X$, in other words, it is the line through X parallel to $A_2 A_3$.

Let M_i be the reflection of L_i in $A_i O$. The angle
$M_2 X L_2 = 2$ angle $O X L_2 = 2$ angle $A_1 A_2 O$ (since $A_1 A_2$ is parallel to $L_2 X$) = angle $A_1 A_2 A_3$, which is the angle between $L_2 X$ and $A_2 A_3$. So $M_2 X$ is parallel to $A_2 A_3$, in other words, M_2 lies on the reflection of $K_2 K_3$ in $L_2 L_3$.

If follows similarly that M_3 lies on the reflection. Similarly, the line $M_1 M_3$ is the reflection of $K_1 K_3$ in $L_1 L_3$, and the line $M_1 M_2$ is the reflection of $K_1 K_2$ in $L_1 L_2$ and hence the triangle formed by the intersections of the three reflections is just $M_1 M_2 M_3$.