42nd International Mathematical Olympiad

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Problems

Each problem is worth seven points.

Problem 1

Let ABC be an acute-angled triangle with circumcentre O. Let P on BC be the foot of the altitude from A.

Suppose that $\angle BCA \geq \angle ABC + 30^\circ$.

Prove that $\angle CAB + \angle COP < 90^\circ$.

Problem 2

Prove that

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1$$

for all positive real numbers a, b and c.

Problem 3

Twenty-one girls and twenty-one boys took part in a mathematical contest.

- Each contestant solved at most six problems.
- For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

Problem 4

Let n be an odd integer greater than 1, and let $k_1, k_2, \ldots, k_n$ be given integers. For each of the $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$, let

$$S(a) = \sum_{i=1}^{n} k_i a_i.$$

Prove that there are two permutations b and c, $b \neq c$, such that $n!$ is a divisor of $S(b) - S(c)$.

Problem 5

In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA.

It is known that $\angle BAC = 60^\circ$ and that $AB + BP = AQ + QB$.

What are the possible angles of triangle ABC?

Problem 6

Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that

$$a c + b d = (b + d + a - c)(b + d - a + c).$$

Prove that $a b + c d$ is not prime.

Problems with Solutions

Problem 1

Let ABC be an acute-angled triangle with circumcentre O. Let P on BC be the foot of the altitude from A.

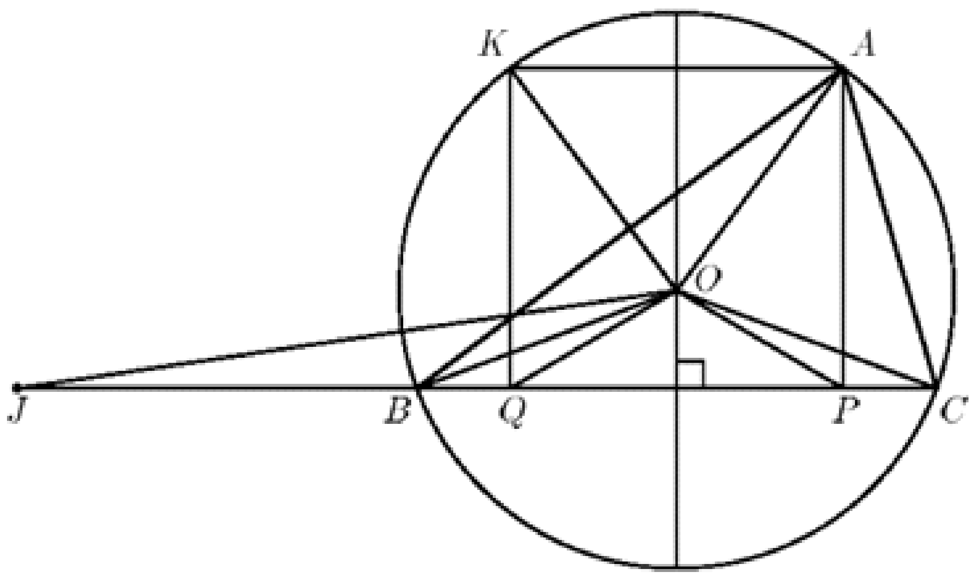
Suppose that $\angle BCA \geq \angle ABC + 30^\circ$.

Prove that $\angle CAB + \angle COP < 90^\circ$.

Solution

■ Solution 1

Let $\alpha = \angle CAB$, $\beta = \angle ABC$, $\gamma = \angle BCA$, and $\delta = \angle COP$. Let K and Q be the reflections of A and P, respectively, across the perpendicular bisector of BC. Let R denote the circumradius of $\triangle ABC$. Then $OA = OB = OC = OK = R$. Furthermore, we have $QP = KA$ because $KQPA$ is a rectangle. Now note that $\angle AOK = \angle AOB - \angle KOB = \angle AOB - \angle AOC = 2\gamma - 2\beta \geq 60^\circ$.



It follows from this and from $OA = OK = R$ that $KA \geq R$ and $QP \geq R$. Therefore, using the Triangle Inequality, we have $OP + R = OQ + OC > QC = QP + PC \geq R + PC$. It follows that $OP > PC$, and hence in $\triangle COP$, $\angle PCO > \delta$. Now since $\alpha = \frac{1}{2} \angle BOC = \frac{1}{2}(180^\circ - 2\angle PCO) = 90^\circ - \angle PCO$, it indeed follows that $\alpha + \delta < 90^\circ$.

■ Solution 2

As in the previous solution, it is enough to show that $OP > PC$. To this end, recall that by the (Extended) Law of Sines, $AB = 2R\sin\gamma$ and $AC = 2R\sin\beta$. Therefore, we have

$$BP - PC = AB\cos\beta - AC\cos\gamma = 2R(\sin\gamma\cos\beta - \sin\beta\cos\gamma) = 2R\sin(\gamma - \beta).$$

It follows from this and from

$$30^\circ \leq \gamma - \beta < \gamma < 90^\circ$$

that $BP - PC \geq R$. Therefore, we obtain that $R + OP = BO + OP > BP \geq R + PC$, from which $OP > OC$, as desired.

■ Solution 3

We first show that $R^2 > CP \cdot CB$. To this end, since $CB = 2R \sin\alpha$ and $CP = AC \cos\gamma = 2R \sin\beta \cos\gamma$, it suffices to show that $\frac{1}{4} > \sin\alpha \sin\beta \cos\gamma$. We note that $1 > \sin\alpha = \sin(\gamma + \beta) = \sin\gamma \cos\beta + \sin\beta \cos\gamma$ and $\frac{1}{2} \leq \sin(\gamma - \beta) = \sin\gamma \cos\beta - \sin\beta \cos\gamma$ since $30^\circ \leq \gamma - \beta < 90^\circ$. It follows that $\frac{1}{4} > \sin\beta \cos\gamma$ and that $\frac{1}{4} > \sin\alpha \sin\beta \cos\gamma$.

Now we choose a point J on BC so that $CJ \cdot CP = R^2$. It follows from this and from $R^2 > CP \cdot CB$ that $CJ > CB$, so that $\angle OBC > \angle OJC$. Since $OC / CJ = PC / CO$ and $\angle JCO = \angle OCP$, we have $\triangle JCO \cong \triangle OCP$ and $\angle OJC = \angle POC = \delta$. It follows that $\delta < \angle OBC = 90^\circ - \alpha$ or $\alpha + \delta < 90^\circ$.

■ Solution 4

On the one hand, as in the third solution, we have $R^2 > CP \cdot CB$. On the other hand, the power of P with respect to the circumcircle of $\triangle ABC$ is $BP \cdot PC = R^2 - OP^2$. From these two equations we find that

$$OP^2 = R^2 - BP \cdot PC > PC \cdot CB - BP \cdot PC = PC^2,$$

from which $OP > PC$. Therefore, as in the first solution, we conclude that $\alpha + \delta < 90^\circ$.

Problem 2

Prove that

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1$$

for all positive real numbers a, b and c.

Solution

First we shall prove that

$$\frac{a}{\sqrt{a^2 + 8bc}} \geq \frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}} + b^{\frac{4}{3}} + c^{\frac{4}{3}}},$$

or equivalently, that

$$\left(a^{\frac{4}{3}} + b^{\frac{4}{3}} + c^{\frac{4}{3}}\right)^2 \geq a^{\frac{2}{3}}(a^2 + 8bc).$$

The AM-GM inequality yields

$$\begin{aligned}\left(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)^2-\left(a^{\frac{4}{3}}\right)^2 &= \left(b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)\left(a^{\frac{4}{3}}+a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)\\ &\geq 2\,b^{\frac{2}{3}}\,c^{\frac{2}{3}}\cdot 4\,a^{\frac{2}{3}}\,b^{\frac{1}{3}}\,c^{\frac{1}{3}}\\ &= 8\,a^{\frac{2}{3}}\,b\,c.\end{aligned}$$

Thus

$$\begin{aligned}\left(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)^2 &\geq \left(a^{\frac{4}{3}}\right)^2+8\,a^{\frac{2}{3}}\,b\,c\\ &= a^{\frac{2}{3}}(a^2+8\,b\,c),\end{aligned}$$

so

$$\frac{a}{\sqrt{a^2+8\,b\,c}} \geq \frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}.$$

Similarly, we have

$$\frac{b}{\sqrt{b^2+8\,c\,a}} \geq \frac{b^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}} \quad \text{and}$$

$$\frac{c}{\sqrt{c^2+8\,a\,b}} \geq \frac{c^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}.$$

Adding these three inequalities yields

$$\frac{a}{\sqrt{a^2+8\,b\,c}}+\frac{b}{\sqrt{b^2+8\,c\,a}}+\frac{c}{\sqrt{c^2+8\,a\,b}} \geq 1.$$

Comment. The proposer conjectures that for any $a, b, c > 0$ and $\lambda \geq 0$, the following inequality holds:

$$\frac{a}{\sqrt{a^2+\lambda\,b\,c}}+\frac{b}{\sqrt{b^2+\lambda\,c\,a}}+\frac{c}{\sqrt{c^2+\lambda\,a\,b}} \geq \frac{3}{\sqrt{1+\lambda}}.$$

Problem 3

Twenty-one girls and twenty-one boys took part in a mathematical contest.

- Each contestant solved at most six problems.
- For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

Solution

■ Solution 1

We introduce the following symbols: G is the set of girls at the competition and B is the set of boys, P is the set of problems, $P(g)$ is the set of problems solved by $g \in G$, and $P(b)$ is the set of problems solved by $b \in B$. Finally, $G(p)$ is the set of girls that solve $p \in P$ and $B(p)$ is the set of boys that solve p. In terms of this notation, we have that for all $g \in G$ and $b \in B$,

$$(a) \quad |P(g)| \leq 6, \quad |P(b)| \leq 6, \qquad (b) \; P(g) \cap P(b) \neq \emptyset.$$

We wish to prove that some $p \in P$ satisfies $|G(p)| \geq 3$ and $|B(p)| \geq 3$. To do this, we shall assume the contrary and reach a contradiction by counting (two ways) all ordered triples (p, q, r) such that $p \in P(g) \cap P(b)$. With $T = \{(p, g, b) : \; p \in P(g) \cap P(b)\}$, condition (b) yields

$$|T| = \sum_{g \in G} \sum_{b \in B} |P(g) \cap P(b)| \geq |G| \cdot |B| = 21^2. \tag{1}$$

Assume that no $p \in P$ satisfies $|G(p)| \geq 3$ and $|B(p)| \geq 3$. We begin by noting that

$$\sum_{p \in P} |G(p)| = \sum_{g \in G} |P(g)| \leq 6|G| \quad \text{and} \quad \sum_{p \in P} |B(p)| \leq 6|B|. \tag{2}$$

(*Note*. The equality in (2) is obtained by a standard double-counting technique: Let $\chi(g, p) = 1$ if g solves p and $\chi(g, p) = 0$ otherwise, and interchange the orders of summation in $\sum_{p \in P} \sum_{g \in G} \chi(g, p)$.) Let

$$P_+ = \{p \in P : \; |G(p)| \geq 3\},$$
$$P_- = \{p \in P : \; |G(p)| \leq 2\}.$$

Claim. $\sum_{p \in P_-} |G(p)| \geq |G|$; thus $\sum_{p \in P_+} |G(p)| \leq 5|G|$. Also $\sum_{p \in P_+} |B(p)| \geq |B|$; thus $\sum_{p \in P_-} |B(b)| \leq 5|B|$.

Proof. Let $g \in G$ be arbitrary. By the Pigeonhole Principle, conditions (a) and (b) imply that g solves some problem p that is solved by at least $\lceil 21/6 \rceil = 4$ boys. By assumption, $|B(p)| \geq 4$ implies that $p \in P_-$, so every girl solves at least one problem in P_-. Thus

$$\sum_{p \in P_-} |G(p)| \geq |G|. \tag{3}$$

In view of (2) and (3) we have

$$\sum_{p \in P_+} |G(p)| = \sum_{p \in P} |G(p)| - \sum_{p \in P_-} |G(p)| \leq 5|G|.$$

Also, each boy solves a problem that is solved by at least four girls, so each boy solves a problem $p \in P_+$. Thus $\sum_{p \in P_+} |B(p)| \geq |B|$, and the calculation proceeds as before using (2). □

Using the claim just established, we find

$$\begin{aligned} |T| &= \sum_{p\in P} |G(p)| \cdot |B(p)| \\ &= \sum_{p\in P_+} |G(p)| \cdot |B(p)| + \sum_{p\in P_-} |G(p)| \cdot |B(p)| \\ &\le 2\sum_{p\in P_+} |G(p)| + 2\sum_{p\in P_-} |B(p)| \\ &\le 10|G| + 10|B| = 20 \cdot 21. \end{aligned}$$

This contradicts (1), so the proof is complete.

■ Solution 2

Let us use some of the notation given in the first solution. Suppose that for every $p \in P$ either $|G(p)| \le 2$ or $|B(p)| \le 2$. For each $p \in P$, color p red if $|G(p)| \le 2$ and otherwise color it black. In this way, if p is red then $|G(p)| \le 2$ and if p is black then $|B(p)| \le 2$. Consider a chessboard with 21 rows, each representing one of the girls, and 21 columns, each representing one of the boys. For each $g \in G$ and $b \in B$, color the square corresponding to (g, b) as follows: pick $p \in P(g) \cap P(b)$ and assign p's color to that square. (By condition (b), there is always an available choice.) By the Pigeonhole Principle, one of the two colors is assigned to at least $\lceil 441/2 \rceil = 221$ squares, and thus some row has at least $\lceil 221/21 \rceil = 11$ black squares or some column has at least 11 red squares.

Suppose the row corresponding to $g \in G$ has at least 11 black squares. Then for each of 11 squares, the black problem that was chosen in assigning the color was solved by at most 2 boys. Thus we account for at least $\lceil 11/2 \rceil = 6$ distinct problems solved by g. In view of condition (a), g solves only these problems. But then at most 12 boys solve a problem also solved by g, in violation of condition (b).

In exactly the same way, a contradiction is reached if we suppose that some column has at least 11 red squares. Hence some $p \in P$ satisfies $|G(p)| \ge 3$ and $|B(p)| \ge 3$.

Problem 4

Let n be an odd integer greater than 1, and let $k_1, k_2, \ldots, k_n$ be given integers. For each of the $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$, let

$$S(a) = \sum_{i=1}^{n} k_i a_i.$$

Prove that there are two permutations b and c, $b \ne c$, such that $n!$ is a divisor of $S(b) - S(c)$.

Solution

Problem 5

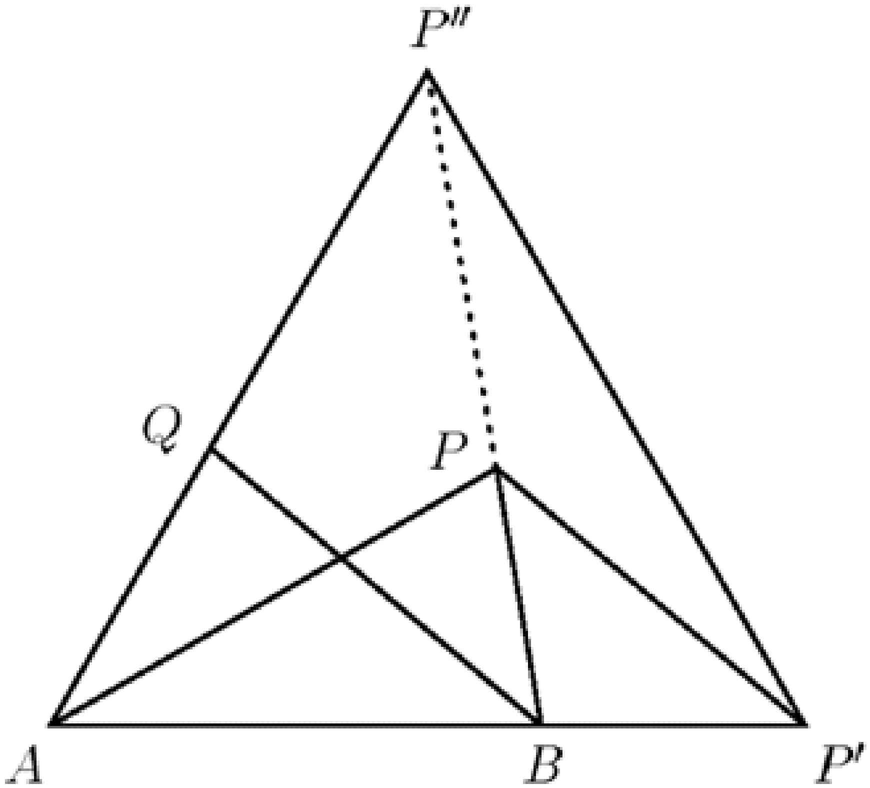
In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA.

It is known that $\angle BAC = 60^\circ$ and that $AB + BP = AQ + QB$.

What are the possible angles of triangle ABC?

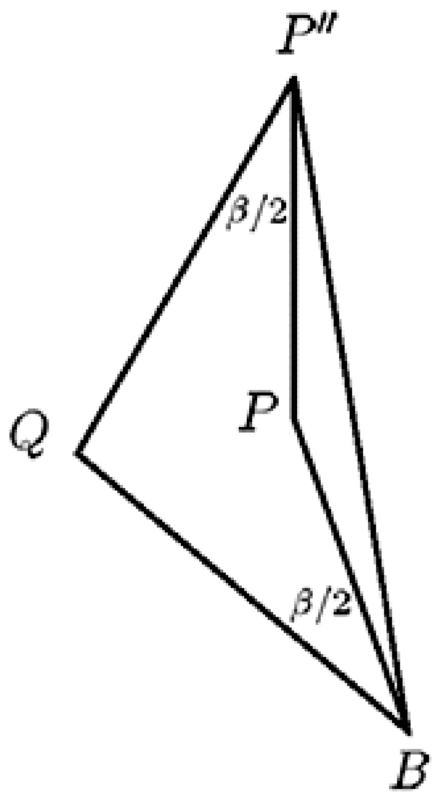
Solution

Denote the angles of ABC by $\alpha = 60^\circ$, β, and γ. Extend AB to P' so that $BP' = BP$, and construct P'' on AQ so that $AP'' = AP'$. Then $BP'P$ is an isosceles triangle with base angle $\beta/2$. Since $AQ + QP'' = AB + BP' = AB + BP = AQ + QB$, it follows that $QP'' = QB$. Since $AP'P''$ is equilateral and AP bisects the angle at A, we have $PP' = PP''$.



Claim. Points B, P, P'' are collinear, so P'' coincides with C.

Proof. Suppose to the contrary that $B P P''$ is a nondegenerate triangle. We have that $\angle P B Q = \angle P P' B = \angle P P'' Q = \beta/2$. Thus the diagram appears as below, or else with P is on the other side of $B P''$. In either case, the assumption that $B P P''$ is nondegenerate leads to $B P = P P'' = P P'$, thus to the conclusion that $B P P'$ is equilateral, and finally to the absurdity $\beta/2 = 60^\circ$ so $\alpha + \beta = 60^\circ + 120^\circ = 180^\circ$.



Thus points B, P, P'' are collinear, and $P'' = C$ as claimed.□

Since triangle BCQ is isosceles, we have $120^\circ - \beta = \gamma = \beta/2$, so $\beta = 80$ and $\gamma = 40^\circ$. Thus ABC is a 60-80-40 degree triangle.

Problem 6

Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that

$$a c + b d = (b + d + a - c)(b + d - a + c).$$

Prove that $a b + c d$ is not prime.

Solution

■ Solution 1

Suppose to the contrary that $a b + c d$ is prime. Note that

$$a b + c d = (a + d) c + (b - c) a = m \cdot \gcd(a + d, b - c)$$

for some positive integer m. By assumption, either $m = 1$ or $g c d(a + d, b - c) = 1$. We consider these alternatives in turn.

Case (i): $m = 1$. Then

$$\begin{aligned}\gcd(a+d, b-c) &= a\,b + c\,d > a\,b + c\,d - (a - b + c + d) \\ &= (a+d)\,(c-1) + (b-c)\,(a+1) \\ &\geq \gcd(a+d, b-c),\end{aligned}$$

which is false.

Case (ii): $\gcd(a+d, b-c) = 1$. Substituting $a\,c + b\,d = (a+d)\,b - (b-c)\,a$ for the left-hand side of $a\,c + b\,d = (b+d+a-c)\,(b+d-a+c)$, we obtain

$$(a+d)\,(a-c-d) = (b-c)\,(b+c+d).$$

In view of this, there exists a positive integer k such that

$$\begin{aligned}a-c-d &= k(b-c), \\ b+c+d &= k(a+d).\end{aligned}$$

Adding these equations, we obtain $a+b = k(a+b-c+d)$ and thus $k(c-d) = (k-1)\,(a+b)$. Recall that $a > b > c > d$. If $k = 1$ then $c = d$, a contradiction. If $k \geq 2$ then

$$2 \geq \frac{k}{k-1} = \frac{a+b}{c-d} > 2,$$

a contradiction.

Since a contradiction is reached in both (i) and (ii), $a\,b + c\,d$ is not prime.

■ Solution 2

The equality $a\,c + b\,d = (b+d+a-c)\,(b+d-a+c)$ is equivalent to

$$a^2 - a\,c + c^2 = b^2 + b\,d + d^2. \tag{1}$$

Let $A\,B\,C\,D$ be the quadrilateral with $A\,B = a$, $B\,C = d$, $C\,D = b$, $A\,D = c$, $\angle B\,A\,D = 60^\circ$, and $\angle B\,C\,D = 120^\circ$. Such a quadrilateral exists in view of (1) and the Law of Cosines; the common value in (1) is $B\,D^2$. Let $\angle A\,B\,C = \alpha$, so that $\angle C\,D\,A = 180^\circ - \alpha$. Applying the Law of Cosines to triangles $A\,B\,C$ and $A\,C\,D$ gives

$$a^2 + d^2 - 2\,a\,d\cos\alpha = A\,C^2 = b^2 + c^2 + 2\,b\,c\cos\alpha.$$

Hence $2\cos\alpha = (a^2 + d^2 - b^2 - c^2)/(a\,d + b\,c)$, and

$$A\,C^2 = a^2 + d^2 - a\,d\,\frac{a^2 + d^2 - b^2 - c^2}{a\,d + b\,c} = \frac{(a\,b + c\,d)\,(a\,c + b\,d)}{a\,d + b\,c.}$$

Because $A\,B\,C\,D$ is cyclic, Ptolemy's Theorem gives

$$(A\,C \cdot B\,D)^2 = (a\,b + c\,d)^2$$

It follows that

$$(a\,c + b\,d)\,(a^2 - a\,c + c^2) = (a\,b + c\,d)\,(a\,d + b\,c). \qquad (2)$$

(*Note*. Also straightforward algebra can be used obtain (2) from (1).) Next observe that

$$a\,b + c\,d > a\,c + b\,d > a\,d + b\,c. \qquad (3)$$

The first inequality follows from $(a - d)\,(b - c) > 0$, and the second from $(a - b)\,(c - d) > 0$.

Now assume that $a\,b + c\,d$ is prime. It then follows from (3) that $a\,b + c\,d$ and $a\,c + b\,d$ are relatively prime. Hence, from (2), it must be true that $a\,c + b\,d$ divides $a\,d + b\,c$. However, this is impossible by (3). Thus $a\,b + c\,d$ must not be prime.

Note. Examples of 4-tuples (a, b, c, d) that satisfy the given conditions are $(21, 18, 14, 1)$ and $(65, 50, 34, 11)$.